

Package ‘ExtremeRisks’

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Title Extreme Risk Measures

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Imports evd, copula, mvtnorm, plot3D, tmvtnorm, pracma

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Description A set of procedures for estimating risks related to extreme events via risk measures such as Expectile, Value-at-Risk, etc. is provided. Estimation methods for univariate independent observations and temporal dependent observations are available. The methodology is extended to the case of independent multidimensional observations. The statistical inference is performed through parametric and non-parametric estimators. Inferential procedures such as confidence intervals, confidence regions and hypothesis testing are obtained by exploiting the asymptotic theory. Adapts the methodologies derived in Padoan and Stupfler (2020) <arxiv:2004.04078>, Padoan and Stupfler (2020) <arxiv:2007.08944>, Daouia et al. (2006) <doi:10.1007/0-387-34471-3>, de Haan et al. (2016) <doi:10.1007/s00780-015-0287-6>.

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dowjones	<i>Negative log-returns of DOW JONES.</i>
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Description

Series of negative log-returns of the U.S. stock market index Dow Jones.

Format

A 8784 * 2 data frame.

Details

From the series of $n = 8785$ closing prices $S_t, t = 1, 2, \dots$, for the Dow Jones stock market index, recorded from January 29, 1985 to December 12, 2019, the series of negative log-returns.

$$X_{t+1} = -\log(S_{t+1}/S_t), \quad 1 \leq t \leq n - 1$$

is available. Hence the dataset (negative log-returns) contains 8784 observations.

EBTailIndex

*Expectile Based Tail Index Estimation***Description**

Computes a point estimate of the tail index based on the Expectile Based (EB) estimator.

Usage

```
EBTailIndex(data, tau, est=NULL)
```

Arguments

data	A vector of $(1 \times n)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
est	A real specifying the estimate of the expectile at the intermediate level tau.

Details

For a dataset data of sample size n , the tail index γ of its (marginal) distribution is estimated using the EB estimator:

$$\hat{\gamma}_n^E = \left(1 + \frac{\hat{F}_n(\tilde{\xi}_{\tau_n})}{1-\tau_n} \right)^{-1},$$

where \hat{F}_n is the empirical survival function of the observations, $\tilde{\xi}_{\tau_n}$ is an estimate of the τ_n -th expectile. The observations can be either independent or temporal dependent. See Padoan and Stupfler (2020) and Daouia et al. (2018) for details.

- The so-called intermediate level tau or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, $\tau_n \in (0, 1)$ is the ratio between the empirical mean distance of the τ_n -th expectile from the smaller observations and the empirical mean distance of the τ_n -th expectile from all the observations. An estimate of τ_n -th expectile is computed and used in turn to estimate γ .
- The value est, if provided, is meant to be an estimate of the τ_n -th expectile which is used to estimate γ . On the contrary, if est=NULL, then the routine EBTailIndex estimate first the τ_n -th expectile and then use it to estimate γ .

Value

An estimate of the tail index γ .

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References

- Padoan A.S. and Stupfler, G. (2020). Extreme expectile estimation for heavy-tailed time series. *arXiv e-prints* arXiv:2004.04078, <https://arxiv.org/abs/2004.04078>.
- Daouia, A., Girard, S. and Stupfler, G. (2018). Estimation of tail risk based on extreme expectiles. *Journal of the Royal Statistical Society: Series B*, **80**, 263-292.

See Also

[HTailIndex](#), [MomTailIndex](#), [MLTailIndex](#),

Examples

```
# Tail index estimation based on the Expectile based estimator obtained with data
# simulated from an AR(1) with 1-dimensional Student-t distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallblock <- 15

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.97

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# tail index estimation
gammaHat <- EBTailIndex(data, tau)
gammaHat
```

estExpectiles

High Expectile Estimation

Description

Computes a point and interval estimate of the expectile at the intermediate level.

Usage

```
estExpectiles(data, tau, method="LAWS", tailest="Hill", var=FALSE, varType="asym-Dep-Adj",
              bigBlock=NULL, smallBlock=NULL, k=NULL, alpha=0.05)
```

Arguments

data	A vector of $(1 \times n)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
method	A string specifying the method used to estimate the expectile. By default <code>est="LAWS"</code> specifies the use of the direct LAWS estimator. See Details .
tailest	A string specifying the type of tail index estimator. By default <code>tailest="Hill"</code> specifies the use of Hill estimator. See Details .
var	If <code>var=TRUE</code> then an estimate of the variance of the expectile estimator is computed.
varType	A string specifying the asymptotic variance to compute. By default <code>varType="asym-Dep-Adj"</code> specifies the variance estimator for serial dependent observations implemented with a suitable adjustment. See Details .
bigBlock	An interger specifying the size of the big-block used to estimaste the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimaste the asymptotic variance. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the expectile at the intermedite level.

Details

For a dataset `data` of sample size n , an estimate of the τ_n -th expectile is computed. Two estimators are available: the so-called direct Least Asymmetrically Weighted Squares (LAWS) and indirect Quantile-Based (QB). The definition of the QB estimator depends on the estimation of the tail index γ . Here, γ is estimated using the Hill estimation (see [HTailIndex](#)) or in alternative using the the expectile based estimator (see [EBTailIndex](#)). The observations can be either independent or temporal dependent. See Section 3.1 in Padoan and Stupfler (2020) for details.

- The so-called intermediate level τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, $\tau_n \in (0, 1)$ is the ratio between N (Numerator) and D (Denominator). Where N is the empirical mean distance of the τ_n -th expectile from the observations smaller than it, and D is the empirical mean distance of τ_n -th expectile from all the observations.
- If `method='LAWS'`, then the expectile at the intermediate level τ_n is estimated applying the direct LAWS estimator. Instead, If `method='QB'` the indirect QB esimator is used to estimate the expectile. See Section 3.1 in Padoan and Stupfler (2020) for details.
- When the expectile is estimated by the indirect QB esimator (`method='QB'`), an estimate of the tail index γ is needed. If `tailest='Hill'` then γ is estimated using the Hill estimator (see also [HTailIndex](#)). If `tailest='ExpBased'` then γ is estimated using the expectile based estimator (see [EBTailIndex](#)). See Section 3.1 in Padoan and Stupfler (2020) for details.

- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, when `method='LAWS'` and `tau=NULL`, k_n specifies by $\tau_n = 1 - k_n/n$ the intermediate level of the expectile. Instead, when `method='QB'`, if `tailest="Hill"` then the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate γ by the Hill estimator and if `tau=NULL` then it also specifies by $\tau_n = 1 - k_n/n$ the confidence level τ_n of the quantile to estimate. Finally, if `tailest="ExpBased"` and `tau=NULL` then it also specifies by $\tau_n = 1 - k_n/n$ the intermediate level expectile based estimator of γ (see [EBTailIndex](#)).
- If `var=TRUE` then the asymptotic variance of the expectile estimator is computed. With independent observations the asymptotic variance is computed by the formula Theorem 3.1 of Padoan and Stupfler (2020). This is achieved through `varType="asym-Ind"`. With serial dependent observations the asymptotic variance is estimated by the formula in Theorem 3.1 of Padoan and Stupfler (2020). This is achieved through `varType="asym-Dep"`. In this latter case the computation of the asymptotic variance is based on the "big blocks separated by small blocks" technique which is a standard tool in time series, see Leadbetter et al. (1986). See also Section C.1 in Appendix of Padoan and Stupfler (2020). The size of the big and small blocks are specified by the parameters `bigblock` and `smallblock`, respectively. Still with serial dependent observations, If `varType="asym-Dep-Adj"`, then the asymptotic variance is estimated using formula (C.79) in Padoan and Stupfler (2020), see Section C.1 of the Appendix for details.
- Given a small value $\alpha \in (0, 1)$ then an asymptotic confidence interval for the τ_n -th expectile, with approximate nominal confidence level $(1 - \alpha)100\%$ is computed. See Sections 3.1 and C.1 in the Appendix of Padoan and Stupfler (2020).

Value

A list with elements:

- `ExpctHat`: a point estimate of the τ_n -th expectile;
- `VarExpHat`: an estimate of the asymptotic variance of the expectile estimator;
- `CIExpct`: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for τ_n -th expectile.

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References

- Padoan A.S. and Stupfler, G. (2020). Extreme expectile estimation for heavy-tailed time series. *arXiv e-prints* arXiv:2004.04078, <https://arxiv.org/abs/2004.04078>.
- Daouia, A., Girard, S. and Stupfler, G. (2018). Estimation of tail risk based on extreme expectiles. *Journal of the Royal Statistical Society: Series B*, **80**, 263-292.
- Leadbetter, M.R., Lindgren, G. and Rootzen, H. (1989). Extremes and related properties of random sequences and processes. *Springer*.

See Also

[HTailIndex](#), [EBTailIndex](#), [predExpectiles](#), [extQuantile](#)

Examples

```

# Extreme expectile estimation at the intermediate level tau obtained with
# 1-dimensional data simulated from an AR(1) with Student-t innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.99

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# High expectile (intermediate level) estimation
expectHat <- estExpectiles(data, tau, var=TRUE, bigBlock=bigBlock, smallBlock=smallBlock)
expectHat$ExpctHat
expectHat$CIEpct

```

estExtLevel

Extreme Level Estimation

Description

Estimates the expectile's extreme level corresponding to a quantile's extreme level.

Usage

```
estExtLevel(alpha_n, data=NULL, gammaHat=NULL, VarGamHat=NULL, tailest="Hill", k=NULL,
            var=FALSE, varType="asym-Dep", bigBlock=NULL, smallBlock=NULL, alpha=0.05)
```

Arguments

alpha_n	A real in $(0, 1)$ specifying the extreme level α_n for the quantile. See Details .
data	A vector of $(1 \times n)$ observations to be used to estimate the tail index in the case it is not provided. By default data=NULL specifies that no data are given.
gammaHat	A real specifying an estimate of the tail index. By default gammaHat=NULL specifies that no estimate is given. See Details .

VarGamHat	A real specifying an estimate of the variance of the tail index estimate. By default VarGamHat=NULL specifies that no estimate is given. See Details .
tailest	A string specifying the type of tail index estimator to be used. By default tailest="Hill" specifies the use of Hill estimator. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
var	If var=TRUE then an estimate of the variance of the extreme level estimator is computed.
varType	A string specifying the asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for serial dependent observations. See Details .
bigBlock	An interger specifying the size of the big-block used to estimaste the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimaste the asymptotic variance. See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the expcile at the intermedite level.

Details

For a given extreme level α_n for the α_n -th quantile, an estimate of the extreme level $\tau'_n(\alpha_n)$ is computed such that $\xi_{\tau'_n(\alpha_n)} = q_{\alpha_n}$. The estimator is defined by

$$\hat{\tau}'_n(\alpha_n) = 1 - (1 - \alpha_n) \frac{\hat{\gamma}_n}{1 - \hat{\gamma}_n}$$

where $\hat{\gamma}_n$ is a consistent estimator of the tail index γ . If a value for the parameter gammaHat is given, then such a value is used to compute $\hat{\tau}'_n$. If gammaHat is NULL and a dataset is provided through the parameter data, then the tail index γ is estimated by a suitable estimator $\hat{\gamma}_n$. See Section 6 in Padoan and Stupfler (2020) for more details.

- If VarGamHat is specified, i.e. the variance of the tail index estimator, then the variance of the extreme level estimator $\hat{\tau}'_n$ is computed by using such value.
- When estimating the tail index, if tailest='Hill' then γ is estimated using the Hill estimator (see also [HTailIndex](#)). If tailest='ML' then γ is estimated using the Maximum Likelihood estimator (see [MLTailIndex](#)). If tailest='ExpBased' then γ is estimated using the expcile based estimator (see [EBTailIndex](#)). If tailest='Moment' then γ is estimated using the moment based estimator (see [MomTailIndex](#)). See Padoan and Stupfler (2020) for details.
- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. Its represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, when tailest="Hill" then the value k_n specifies the number of k+1 larger order statistics to be used to estimate γ by the Hill estimator. See [MLTailIndex](#), [EBTailIndex](#) and [MomTailIndex](#) for the other estimators.
- If var=TRUE then the asymptotic variance of the extreme level estimator is computed by applying the delta method, i.e.

$$Var(\tau'_n) = Var(\hat{\gamma}_n) * (\alpha_n - 1)^2 / (1 - \hat{\gamma}_n)^4$$
 where $Var(\hat{\gamma}_n)$ is provided by VarGamHat or is estimated when esitimating the tail index through tailest='Hill' and tailest='ML'. See [HTailIndex](#) and [MLTailIndex](#) for details on how the variance is computed.
- Given a small value $\alpha \in (0, 1)$ then an asymptotic confidence interval for the extreme level, $\tau'_n(\alpha_n)$, with approximate nominal confidence level $(1 - \alpha)100\%$ is computed.

Value

A list with elements:

- tauHat: an estimate of the extreme level τ'_n ;
- tauVar: an estimate of the asymptotic variance of the extreme level estimator $\hat{\tau}'_n(\alpha_n)$;
- tauCI: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for the extreme level $\tau'_n(\alpha_n)$.

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References

- Padoan A.S. and Stupfler, G. (2020). Extreme expectile estimation for heavy-tailed time series. *arXiv e-prints* arXiv:2004.04078, <https://arxiv.org/abs/2004.04078>.
- Daouia, A., Girard, S. and Stupfler, G. (2018). Estimation of tail risk based on extreme expectiles. *Journal of the Royal Statistical Society: Series B*, **80**, 263-292.

See Also

[estExpectiles](#), [predExpectiles](#), [extQuantile](#)

Examples

```
# Extreme level estimation for a given quantile's extreme level alpha_n
# obtained with 1-dimensional data simulated from an AR(1) with Student-t innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# quantile's extreme level
alpha_n <- 0.999

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)
```

```
# expectile's extreme level estimation
tau1Hat <- estExtLevel(alpha_n, data, var=TRUE, k=150, bigBlock=bigBlock,
                      smallBlock=smallBlock)

tau1Hat
```

estMultiExpectiles *Multidimensional High Expectile Estimation*

Description

Computes point estimates and $(1 - \alpha)100\%$ confidence regions for d-dimensional expectiles at the intermediate level.

Usage

```
estMultiExpectiles(data, tau, method="LAWS", tailest="Hill", var=FALSE,
                  varType="asym-Ind-Adj", k=NULL, alpha=0.05, plot=FALSE)
```

Arguments

data	A matrix of $(n \times d)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
method	A string specifying the method used to estimate the expectile. By default est="LAWS" specifies the use of the direct LAWS estimator. See Details .
tailest	A string specifying the type of tail index estimator. By default tailest="Hill" specifies the use of Hill estimator. See Details .
var	If var=TRUE then an estimate of the variance of the expectile estimator is computed.
varType	A string specifying the asymptotic variance-covariance matrix to compute. By default varType="asym-Ind-Adj" specifies that the variance-covariance matrix is computed assuming dependent variables and exploiting a suitable adjustment. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence region for the d-dimensional expectile at the intermediate level.
plot	A logical value. By default plot=FALSE specifies that no graphical representation of the estimates is not provided. See Details .

Details

For a dataset data of d-dimensional observations and sample size n , an estimate of the τ_n -th d-dimensional is computed. Two estimators are available: the so-called direct Least Asymmetrically Weighted Squares (LAWS) and indirect Quantile-Based (QB). The QB estimator depends on the estimation of the d-dimensional tail index γ . Here, γ is estimated using the Hill estimator (see [MultiHTailIndex](#)). The data are regarded as d-dimensional temporal independent observations coming from dependent variables. See Padoan and Stupfler (2020) for details.

- The so-called intermediate level τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, for each individual marginal distribution $\tau_n \in (0, 1)$ is the ratio between N (Numerator) and D (Denominator). Where N is the empirical mean distance of the τ_n -th expectile from the observations smaller than it, and D is the empirical mean distance of τ_n -th expectile from all the observations.
- If method='LAWS', then the expectile at the intermediate level τ_n is estimated applying the direct LAWS estimator. Instead, If method='QB' the indirect QB estimator is used to estimate the expectile. See Section 2.1 in Padoan and Stupfler (2020) for details.
- When the expectile is estimated by the indirect QB estimator (method='QB'), an estimate of the d-dimensional tail index γ is needed. Here the d-dimensional tail index γ is estimated using the d-dimensional Hill estimator (tailest='Hill', see [MultiHTailIndex](#)). This is the only available option so far (soon more results will be available).
- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, for each marginal distribution, when method='LAWS' and tau=NULL, k_n specifies by $\tau_n = 1 - k_n/n$ the intermediate level of the expectile. Instead, for each marginal distribution, when method='QB', then the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate γ by the Hill estimator and if tau=NULL then it also specifies by $\tau_n = 1 - k_n/n$ the confidence level τ_n of the quantile to estimate.
- If var=TRUE then an estimate of the asymptotic variance-covariance matrix of the d-dimensional expectile estimator is computed. If the data are regarded as d-dimensional temporal independent observations coming from dependent variables. Then, the asymptotic variance-covariance matrix is estimated by the formulas in section 3.1 of Padoan and Stupfler (2020). In particular, the variance-covariance matrix is computed exploiting the asymptotic behaviour of the relative expectile estimator appropriately normalized and using a suitable adjustment. This is achieved through varType="asym-Ind-Adj". The data can also be regarded as coded-dimensional temporal independent observations coming from independent variables. In this case the asymptotic variance-covariance matrix is diagonal and is also computed exploiting the formulas in section 3.1 of Padoan and Stupfler (2020). This is achieved through varType="asym-Ind".
- Given a small value $\alpha \in (0, 1)$ then an asymptotic confidence region for the τ_n -th expectile, with approximate nominal confidence level $(1 - \alpha)100\%$ is computed. In particular, a "symmetric" confidence regions is computed exploiting the asymptotic behaviour of the relative expectile estimator appropriately normalized. See Sections 3.1 of Padoan and Stupfler (2020) for detailed.
- If plot=TRUE then a graphical representation of the estimates is not provided.

Value

A list with elements:

- ExpctHat: an point estimate of the τ_n -th d-dimensional expectile;
- biasTerm: an point estimate of the bias term of the estimated expectile;
- VarCovEHat: an estimate of the asymptotic variance of the expectile estimator;
- EstConReg: an estimate of the approximate $(1 - \alpha)100\%$ confidence region for τ_n -th d-dimensional expectile.

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References

Padoan A.S. and Stupfler, G. (2020). Joint inference on extreme expectiles for multivariate heavy-tailed distributions. *arXiv e-prints* arXiv:2007.08944, <https://arxiv.org/abs/2007.08944>

See Also

[MultiHTailIndex](#), [predMultiExpectiles](#), [extMultiQuantile](#)

Examples

```
# Extreme expectile estimation at the intermediate level tau obtained with
# d-dimensional observations simulated from a joint distribution with
# a Gumbel copula and equal Frechet marginal distributions.
library(plot3D)
library(copula)
library(evd)

# distributional setting
copula <- "Gumbel"
dist <- "Frechet"

# parameter setting
dep <- 3
dim <- 3
scale <- rep(1, dim)
shape <- rep(3, dim)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# Intermediate level (or sample tail probability 1-tau)
tau <- .95

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
# marginals distributions and a Gumbel copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# High d-dimensional expectile (intermediate level) estimation
expectHat <- estMultiExpectiles(data, tau, var=TRUE)
expectHat$ExpctHat
expectHat$VarCovEHat
# run the following command to see the graphical representation

expectHat <- estMultiExpectiles(data, tau, var=TRUE, plot=TRUE)
```

 expectiles

Expectile Computation

Description

Computes the true expectile for some families of parametric models.

Usage

```
expectiles(par, tau, tsDist="gPareto", tsType="IID", trueMethod="true",
           estMethod="LAWS", nrep=1e+05, ndata=1e+06, burnin=1e+03)
```

Arguments

par	A vector of $(1 \times p)$ parameters of the time series parametric family. See Details .
tau	A real in $(0, 1)$ specifying the level τ of the expectile to be computed. See Details .
tsDist	A string specifying the parametric family of the innovations distribution. By default <code>tsDist="gPareto"</code> specifies a Pareto family of distributions. See Details .
tsType	A string specifying the type of time series. By default <code>tsType="IID"</code> specifies a sequence of independent and identically distributed random variables. See Details .
trueMethod	A string specifying the method used to computed the expecile. By default <code>trueMethod="true"</code> specifies that the true analytical expression to computed the expectile is used. See Details .
estMethod	A string specifying the method used to estimate the expecile. By default <code>est="LAWS"</code> specifies the use of the direct LAWS estimator. See Details .
nrep	A positive interger specifying the number of simulations to use for computing an approximation of the expectile. See Details .
ndata	A positive interger specifying the number of observations to genreated for each simulation. See Details .
burnin	A positive interger specifying the number of initial observations to discard from the simulated sample.

Details

For a parametric family of time series models or a parametric family of distributions (for the case of independent observations) the τ -th expectile (or expectile of level tau) is computed.

- There are two methods to compute the τ -th expectile. For the Generalised Pareto and Student- t parametric families of distributions, the analytical epxression of the expectile is available. This is used to compute the τ -th expectile if the parameter `trueMethod="true"` is specified. For most of parametric family of distributions or parametric families of time series models the analytical epxression of the expectile is not available. In this case an approximate value of the τ -th expectile is computed via a Monte Carlo method if the parameter

`trueMethod=="approx"` is specified. In particular, `n` data observations from a family of time series models (e.g. `tsType="AR"` and `tsDist="studentT"`) or a sequence of independent and identically distributed random variables with common family of distributions (e.g. `tsType="IID"` and `tsDist="gPareto"`) are simulated `nrep` times. For each simulation the τ -th expectile is estimate by the estimation method specified by `estMethod`. The mean of such estimate provides an approximate value of the τ -th expectile. The available estimator to esitmate the expecile are the direct LAWS (`estMethod="LAWS"`) and the indirect QB (`estMethod="QB"`), see [estExpectiles](#) for details. The available families of distributions are: Generalised Pareto (`tsDist="gPareto"`), Student- t (`tsDist="studentT"`) and Frechet (`tsDist="Frechet"`). The available classes of time series with parametric innovations families of distributions are specified in [rtimeseries](#).

Value

The τ -th expectile.

Author(s)

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References

Padoan A.S. and Stupfler, G. (2020). Extreme expectile estimation for heavy-tailed time series. *arXiv e-prints* arXiv:2004.04078, <https://arxiv.org/abs/2004.04078>.

See Also

[rtimeseries](#)

Examples

```
# Derivation of the true tau-th expectile for the Pareto distribution
# via accurate simulation

# parameter value
par <- c(1, 0.3)

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.99

trueExp <- expectiles(par, tau)
trueExp

# tau-th expectile of the AR(1) with Student-t innovations
tsDist <- "studentT"
tsType <- "AR"

# Approximation via Monte Carlo methods
trueMethod <- "approx"
```

```

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.99

trueExp <- expectiles(par, tau, tsDist, tsType, trueMethod)
trueExp

```

ExpectMES

Marginal Expected Shortfall Expectile Based Estimation

Description

Computes a point and interval estimate of the Marginal Expected Shortfall (MES) using an expectile based approach.

Usage

```
ExpectMES(data, tau, tau1, method="LAWS", var=FALSE, varType="asym-Dep", bias=FALSE,
          bigBlock=NULL, smallBlock=NULL, k=NULL, alpha_n=NULL, alpha=0.05)
```

Arguments

data	A vector of $(1 \times n)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
method	A string specifying the method used to estimate the expectile. By default est="LAWS" specifies the use of the LAWS based estimator. See Details .
var	If var=TRUE then an estimate of the asymptotic variance of the MES estimator is computed.
varType	A string specifying the type of asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for serial dependent observations. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
bigBlock	An interger specifying the size of the big-block used to estimate the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimate the asymptotic variance. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .

alpha_n	A real in $(0, 1)$ specifying the quantile's extreme level to be use in order to estimate the expectile's extreme level.
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the expectile at the intermedite level.

Details

For a dataset data of sample size n , an estimate of the τ'_n -th MES is computed. The estimation of the MES at the extreme level τ'_n is indeed meant to be a prediction. Two estimators are available: the so-called Least Asymmetrically Weighted Squares (LAWS) based estimator and the Quantile-Based (QB) estimator. The definition of both estimators depends on the estimation of the tail index γ . Here, γ is estimated using the Hill estimation (see [HTailIndex](#) for details). The observations can be either independent or temporal dependent. See Section 4 in Padoan and Stupfler (2020) for details.

- The so-called intermediate level τ_n or τ'_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. See [predExpectiles](#) for details.
- The so-called extreme level τ_1 or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. See [predExpectiles](#) for details.
- When method='LAWS', then the τ'_n -th MES is estimated using the LAWS based estimator. When method='QB', the expectile is instead estimated using the QB estimator. See Section 4 in Padoan and Stupfler (2020) and in particular Corollary 4.3 and 4.4 for details. The definition of both estimators depend on the estimation of the tail index γ . In particular, the tail index γ is estimated using the Hill estimator (see [HTailIndex](#)).
- If var=TRUE then an estimate of the asymptotic variance of the τ'_n -th MES is computed. Notice that the estimation of the asymptotic variance is **only available** when γ is estimated using the Hill estimator (see [HTailIndex](#)). With independent observations the asymptotic variance is estimated by $\hat{\gamma}^2$, see Corollary 4.3 in Padoan and Stupfler (2020). This is achieved through varType="asym-Ind". With serial dependent observations the asymptotic variance is estimated by the formula in Corollary 4.3 of Padoan and Stupfler (2020). This is achieved through varType="asym-Dep". See Section 4 and 5 in Padoan and Stupfler (2020) for details. In this latter case the computation of the serial dependence is based on the "big blocks separated by small blocks" technique which is a standard tools in time series, see e.g. Leadbetter et al. (1986). The size of the big and small blocks are specified by the parameters bigBlock and smallBlock, respectively.
- If bias=TRUE then γ is estimated using formula (4.2) of Haan et al. (2016). This is used by the LAWS and QB estimators. Furthermore, the τ'_n -th quantile is estimated using the formula in page 330 of de Haan et al. (2016). This provides a bias corrected version of the Weissman estimator. This is used by the QB estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. Instead, for simplicity the asymptotic variance is estimated by the formula in Corollary 3.8, with serial dependent observations, and $\hat{\gamma}^2$ with independent observation (see e.g. de Drees 2000, for the details).
- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, when tau=NULL and method='LAWS', then $\tau_n = 1 - k_n/n$ is the intermediate level of the expectile to be stimulated. k_n also specifies the number of $k+1$ larger order statistics used in the definition of the Hill estimator (see [HTailIndex](#) for detail). Differently, When tau=NULL and method='QB', then $\tau_n = 1 - k_n/n$ is the intermediate level of the quantile to be stimulated.

- If the quantile's extreme level is provided by `alpha_n`, then expectile's extreme level τ_n' is replaced by $\tau_n'(\alpha_n)$ which is estimated by the method described in Section 6 of Padoan and Stupfler (2020). See `estExtLevel` for details.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence interval for τ_n' -th expectile, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence intervals are computed exploiting formula in Corollary 4.3, 4.4 and Theorem 6.2 of Padoan and Stupfler (2020) and (46) in Drees (2003). See Sections 4-6 in Padoan and Stupfler (2020) for details. When `bias=TRUE` confidence intervals are computed in the same way but after correcting the tail index estimate by an estimate of the bias term, see formula (4.2) in de Haan et al. (2016) for details.

Value

A list with elements:

- `HatXMES`: an estimate of the τ_n' -th expectile based MES;
- `VarHatXMES`: an estimate of the asymptotic variance of the expectile based MES estimator;
- `CIHatXMES`: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for τ_n' -th MES.

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References

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- Daouia, A., Girard, S. and Stupfler, G. (2018). Estimation of tail risk based on extreme expectiles. *Journal of the Royal Statistical Society: Series B*, **80**, 263-292.
- de Haan, L., Mercadier, C. and Zhou, C. (2016). Adapting extreme value statistics to financial time series: dealing with bias and serial dependence. *Finance and Stochastics*, **20**, 321-354.
- Drees, H. (2003). Extreme quantile estimation for dependent data, with applications to finance. *Bernoulli*, **9**, 617-657.
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See Also

[QuantMES](#), [HTailIndex](#), [predExpectiles](#), [extQuantile](#)

Examples

```

# Marginl Expected Shortfall expectile based estimation at the extreme level
# obtained with 2-dimensional data simulated from an AR(1) with bivariate
# Student-t distributed innovations

tsDist <- "AStudentT"
tsType <- "AR"
tsCopula <- "studentT"

# parameter setting
corr <- 0.8
dep <- 0.8
df <- 3
par <- list(corr=corr, dep=dep, df=df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# quantile's extreme level
alpha_n <- 0.999

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rbtimeseries(ndata, tsDist, tsType, tsCopula, par)

# Extreme MES expectile based estimation
MESHat <- ExpectMES(data, NULL, NULL, var=TRUE, k=150, bigBlock=bigBlock,
                    smallBlock=smallBlock, alpha_n=alpha_n)
MESHat

```

extMultiQuantile	<i>Multidimensional Value-at-Risk (VaR) or Extreme Quantile (EQ) Estimation</i>
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Description

Computes point estimates and $(1 - \alpha)100\%$ confidence regions for d-dimensional VaR based on the Weissman estimator.

Usage

```

extMultiQuantile(data, tau, tau1, var=FALSE, varType="asym-Ind-Log", bias=FALSE,
                 k=NULL, alpha=0.05, plot=FALSE)

```

Arguments

data	A matrix of $(n \times d)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
var	If var=TRUE then an estimate of the asymptotic variance-covariance matrix of the d-dimensional VaR estimator is computed.
varType	A string specifying the type of asymptotic variance-covariance matrix to compute. By default varType="asym-Ind-Log" specifies that the variance-covariance matrix is obtained assuming dependent variables and exploiting the logarithmic scale. See Details .
bias	A logical value. By default biast=FALSE specifies that no bias correction is computed. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence region for the d-dimensional VaR.
plot	A logical value. By default plot=FALSE specifies that no graphical representation of the estimates is not provided. See Details .

Details

For a dataset data of d-dimensional observations and sample size n , the VaR or EQ, corresponding to the extreme level tau1, is computed by applying the d-dimensional Weissman estimator. The definition of the Weissman estimator depends on the estimation of the d-dimensional tail index γ . Here, γ is estimated using the Hill estimation (see [MultiHTailIndex](#)). The data are regarded as d-dimensional temporal independent observations coming from dependent variables. See Padoan and Stupfler (2020) for details.

- The so-called intermediate level tau or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, for each variable, $(1 - \tau_n) \in (0, 1)$ is a small proportion of observations in the observed data sample that exceed the τ_n -th empirical quantile. Such proportion of observations is used to estimate the individual τ_n -th quantile and tail index γ .
- The so-called extreme level tau1 or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. For each variable, the value $(1 - \tau'_n) \in (0, 1)$ is meant to be a small tail probability such that $(1 - \tau'_n) = 1/n$ or $(1 - \tau'_n) < 1/n$. It is also assumed that $n(1 - \tau'_n) \rightarrow C$ as $n \rightarrow \infty$, where C is a positive finite constant. The value C is the expected number of exceedances of the individual τ'_n -th quantile. Typically, $C \in (0, 1)$ which means that it is expected that there are no observations in a data sample exceeding the individual quantile of level $(1 - \tau'_n)$.
- If var=TRUE then an estimate of the asymptotic variance-covariance matrix of the τ'_n -th d-dimensional quantile is computed. The data are regarded as temporal independent observations coming from dependent variables. The asymptotic variance-covariance matrix is estimated exploiting the formula in Section 5 of Padoan and Stupfler (2020). In particular, the variance-covariance matrix is computed exploiting the asymptotic behaviour of the normalized quantile estimator which is expressed in logarithmic scale. This is achieved through varType="asym-Ind-Log". If varType="asym-Ind" then the variance-covariance matrix is computed exploiting the asymptotic behaviour of the d-dimensional relative quantile estimator appropriately normalized (see formula in Section 5 of Padoan and Stupfler (2020)).

- If `bias=TRUE` then an estimate of each individual τ'_n -th quantile is estimated using the formula in page 330 of de Haan et al. (2016), which provides a bias corrected version of the Weissman estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. For simplicity standard the variance-covariance matrix is still computed using formula in Section 5 of Padoan and Stupfler (2020).
- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, for each marginal distribution, the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate the individual τ_n -th empirical quantile and individual tail index γ_j for $j = 1, \dots, d$. The intermediate level τ_n can be seen defined as $\tau_n = 1 - k_n/n$.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence region for τ'_n -th d-dimensional quantile, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence regions are computed exploiting the asymptotic behaviour of the normalized quantile estimator in logarithmic scale. This is an "asymmetric" region and it is achieved through `varType="asym-Ind-Log"`. A "symmetric" region is obtained exploiting the asymptotic behaviour of the relative quantile estimator appropriately normalized, see formula in Section 5 of Padoan and Stupfler (2020). This is achieved through `varType="asym-Ind"`.
- If `plot=TRUE` then a graphical representation of the estimates is not provided.

Value

A list with elements:

- `ExtQHat`: an estimate of the d-dimensional VaR or τ'_n -th d-dimensional quantile;
- `VarCovExQHat`: an estimate of the asymptotic variance-covariance of the d-dimensional VaR estimator;
- `EstConReg`: an estimate of the approximate $(1 - \alpha)100\%$ confidence regions for the d-dimensional VaR.

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References

- Padoan A.S. and Stupfler, G. (2020). Joint inference on extreme expectiles for multivariate heavy-tailed distributions. *arXiv e-prints* arXiv:2007.08944, <https://arxiv.org/abs/2007.08944>
- de Haan, L., Mercadier, C. and Zhou, C. (2016). Adapting extreme value statistics to financial time series: dealing with bias and serial dependence. *Finance and Stochastics*, **20**, 321-354.
- de Haan, L. and Ferreira, A. (2006). Extreme Value Theory: An Introduction. *Springer-Verlag*, New York.

See Also

[MultiHTailIndex](#), [estMultiExpectiles](#), [predMultiExpectiles](#)

Examples

```
# Extreme quantile estimation at the extreme level tau1 obtained with
# d-dimensional observations simulated from a joint distribution with
# a Gumbel copula and equal Frechet marginal distributions.
library(plot3D)
library(copula)
library(efd)

# distributional setting
copula <- "Gumbel"
dist <- "Frechet"

# parameter setting
dep <- 3
dim <- 3
scale <- rep(1, dim)
shape <- rep(3, dim)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.95
# Extreme level (or tail probability 1-tau1 of unobserved quantile)
tau1 <- 0.9995

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
# marginals distributions and a Gumbel copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# High d-dimensional expectile (intermediate level) estimation
extQHat <- extMultiQuantile(data, tau, tau1, TRUE)

extQHat$ExtQHat
extQHat$VarCovExQHat
# run the following command to see the graphical representation

extQHat <- extMultiQuantile(data, tau, tau1, TRUE, plot=TRUE)
```

extQuantile

Value-at-Risk (VaR) or Extreme Quantile (EQ) Estimation

Description

Computes a point and interval estimate of the VaR based on the Weissman estimator.

Usage

```
extQuantile(data, tau, tau1, var=FALSE, varType="asym-Dep", bias=FALSE, bigBlock=NULL,
            smallBlock=NULL, k=NULL, alpha=0.05)
```

Arguments

data	A vector of $(1 \times n)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
var	If var=TRUE then an estimate of the asymptotic variance of the VaR estimator is computed.
varType	A string specifying the type of asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for serial dependent observations. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
bigBlock	An interger specifying the size of the big-block used to estimate the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimate the asymptotic variance. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the VaR.

Details

For a dataset data of sample size n , the VaR or EQ, corresponding to the extreme level tau1, is computed by applying the Weissman estimator. The definition of the Weissman estimator depends on the estimation of the tail index γ . Here, γ is estimated using the Hill estimation (see [HTailIndex](#)). The observations can be either independent or temporal dependent (see e.g. de Haan and Ferreira 2006; Drees 2003; de Haan et al. 2016 for details).

- The so-called intermediate level tau or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, $(1 - \tau_n) \in (0, 1)$ is a small proportion of observations in the observed data sample that exceed the τ_n -th empirical quantile. Such proportion of observations is used to estimate the τ_n -th quantile and γ .
- The so-called extreme level tau1 or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. The value $(1 - \tau'_n) \in (0, 1)$ is meant to be a small tail probability such that $(1 - \tau'_n) = 1/n$ or $(1 - \tau'_n) < 1/n$. It is also assumed that $n(1 - \tau'_n) \rightarrow C$ as $n \rightarrow \infty$, where C is a positive finite constant. The value C is the expected number of exceedances of the τ'_n -th quantile. Typically, $C \in (0, 1)$ which means that it is expected that there are no observations in a data sample exceeding the quantile of level $(1 - \tau'_n)$.
- If var=TRUE then an estimate of the asymptotic variance of the τ'_n -th quantile is computed. With independent observations the asymptotic variance is estimated by the formula $\hat{\gamma}^2$ (see e.g. de Drees 2000, 2003, for details). This is achieved through varType="asym-Ind". With

serial dependent data the asymptotic variance is estimated by the formula in 1288 in Drees (2000). This is achieved through `varType="asym-Dep"`. In this latter case the computation of the serial dependence is based on the "big blocks separated by small blocks" technique which is a standard tool in time series, see e.g. Leadbetter et al. (1986). The size of the big and small blocks are specified by the parameters `bigBlock` and `smallBlock`, respectively. With serial dependent data the asymptotic variance can also be estimated by formula (32) of Drees (2003). This is achieved through `varType="asym-Alt-Dep"`.

- If `bias=TRUE` then an estimate of the τ'_n -th quantile is computed using the formula in page 330 of de Haan et al. (2016), which provides a bias corrected version of the Weissman estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. Instead, for simplicity standard formula in Drees (2000) page 1288 is used.
- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate the τ_n -th empirical quantile and γ . The intermediate level τ_n can be seen defined as $\tau_n = 1 - k_n/n$.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence interval for τ'_n -th quantile, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence intervals are computed exploiting the formulas (33) and (46) of Drees (2003). When `bias=TRUE` confidence intervals are computed in the same way but after correcting the tail index estimate by an estimate of the bias term, see formula (4.2) in de Haan et al. (2016) for details. Furthermore, in this case with serial dependent data the asymptotic variance is estimated using the formula in Drees (2000) page 1288.

Value

A list with elements:

- `ExtQHat`: an estimate of the VaR or τ'_n -th quantile;
- `VarExQHat`: an estimate of the asymptotic variance of the VaR estimator;
- `CIExtQ`: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for the VaR.

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β -mixing random variables. *Annals of Applied Probability*, **10**, 1274-1301.

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See Also

[HTailIndex](#), [EBTailIndex](#), [estExpectiles](#)

Examples

```
# Extreme quantile estimation at the level tau1 obtained with 1-dimensional data
# simulated from an AR(1) with univariate Student-t distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.97
# Extreme level (or tail probability 1-tau1 of unobserved quantile)
tau1 <- 0.9995

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# VaR (extreme quantile) estimation
extQHat1 <- extQuantile(data, tau, tau1, TRUE, bigBlock=bigBlock, smallBlock=smallBlock)
extQHat1$ExtQHat
extQHat1$CIExtQ

# VaR (extreme quantile) estimation with bias correction
extQHat2 <- extQuantile(data, tau, tau1, TRUE, bias=TRUE, bigBlock=bigBlock, smallBlock=smallBlock)
extQHat2$ExtQHat
extQHat2$CIExtQ
```

HTailIndex

*Hill Tail Index Estimation***Description**

Computes a point and interval estimate of the tail index based on the Hill's estimator.

Usage

```
HTailIndex(data, k, var=FALSE, varType="asym-Dep", bias=FALSE, bigBlock=NULL,
            smallBlock=NULL, alpha=0.05)
```

Arguments

data	A vector of $(1 \times n)$ observations.
k	An integer specifying the value of the intermediate sequence k_n . See Details .
var	If var=TRUE then an estimate of the variance of the tail index estimator is computed.
varType	A string specifying the asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for serial dependent observations. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
bigBlock	An interger specifying the size of the big-block used to estimaste the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimaste the asymptotic variance. See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the tail index.

Details

For a dataset data of sample size n , the tail index γ of its (marginal) distribution is computed by applying the Hill estimator. The observations can be either independent or temporal dependent.

- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. Its represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate γ .
- If var=TRUE then an estimate of the asymptotic variance of the Hill estimator is computed. With independent observations the asymptotic variance is estimated by the formula $\hat{\gamma}^2$, see Theorem 3.2.5 of de Haan and Ferreira (2006). This is achieved through varType="asym-Ind". With serial dependent observations the asymptotic variance is estimated by the formula in 1288 in Drees (2000). This is achieved through varType="asym-Dep". In this latter case the serial dependence is estimated by exploiting the "big blocks seperated by small blocks" technique which is a standard tools in time series, see Leadbetter et al. (1986). See also formula (11) in Drees (2003). The size of the big and small blocks are specified by the parameters bigBlock and smallBlock, respectively.

- If `bias=TRUE` then an estimate of the bias term of the Hill estimator is computed implementing using formula (4.2) in de Haan et al. (2016). However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.1. Instead for simplicity standard formulas have been used (see de Haan and Ferreira 2006 Theorem 3.2.5 and Drees 2000 page 1288).
- Given a small value $\alpha \in (0,1)$ then an estimate of an asymptotic confidence interval for γ , with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence intervals are computed exploiting the formulas in de Haan and Ferreira (2006) Theorem 3.2.5 and Drees (2000) page 1288. When `biast=TRUE` the confidence intervals are computed in the same way but after correcting the tail index estimate by an estimate of the bias term, see formula (4.2) in de Haan et al. (2016) for details.

Value

A list with elements:

- `gammaHat`: an estimate of tail index γ ;
- `VarGamHat`: an estimate of the asymptotic variance of the Hill estimator;
- `BiasGamHat`: an estimate of bias term of the Hill estimator;
- `AdjExtQHat`: the adjustment to correct the Weissman estimator of an extreme quantile.

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References

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See Also

[MLTailIndex](#), [MomTailIndex](#), [EBTailIndex](#)

Examples

```

# Tail index estimation based on the Hill estimator obtained with
# 1-dimensional data simulated from an AR(1) with univariate Student-t
# distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# Number of larger order statistics
k <- 150

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# tail index estimation
gammaHat1 <- HTailIndex(data, k, TRUE, bigBlock=bigBlock, smallBlock=smallBlock)
gammaHat1$gammaHat
gammaHat1$CIgamHat

# tail index estimation with bias correction
gammaHat2 <- HTailIndex(data, 2*k, TRUE, bias=TRUE, bigBlock=bigBlock, smallBlock=smallBlock)
gammaHat2$gammaHat-gammaHat2$BiasGamHat
gammaHat2$CIgamHat

```

HypoTesting

Wald-Type Hypothesis Testing

Description

Wald-type hypothesis tes for testing equality of high or extreme expectiles and quantiles

Usage

```

HypoTesting(data, tau, tau1=NULL, type="ExpectRisks", level="extreme",
            method="LAWS", bias=FALSE, k=NULL, alpha=0.05)

```

Arguments

data	A matrix of $(n \times d)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
type	A string specifying the type of test. By default type="ExpectRisks" specifies the test for testing the equality of expectiles. See Details .
level	A string specifying the level of the expectile. This make sense when type="ExpectRisks". By default level="extreme" specifies that the test concerns expectiles at the extreme level. See Details .
method	A string specifying the method used to estimate the expectile. By default est="LAWS" specifies the use of the LAWS based estimator. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the significance level of the test.

Details

With a dataset data of d-dimensional observations and sample size n , a Wald-type hypothesis testing is performed in order to check whether there is empirical evidence against the null hypothesis. The null hypothesis concerns the equality among the expectiles or quantiles or tail indices of the marginal distributions. The three tests depend on the estimation of the d-dimensional tail index γ . Here, γ is estimated using the Hill estimation (see [MultiHTailIndex](#) for details). The data are regarded as d-dimensional temporal independent observations coming from dependent variables. See Padoan and Stupfler (2020) for details.

- The so-called intermediate level tau or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, for each marginal distribution, $\tau_n \in (0, 1)$ is the ratio between N (Numerator) and D (Denominator). Where N is the empirical mean distance of the τ_n -th expectile from the observations smaller than it, and D is the empirical mean distance of τ_n -th expectile from all the observations.
- The so-called extreme level tau1 or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. For each marginal distribution, the value $(1 - \tau'_n) \in (0, 1)$ is meant to be a small tail probability such that $(1 - \tau'_n) = 1/n$ or $(1 - \tau'_n) < 1/n$. It is also assumed that $n(1 - \tau'_n) \rightarrow C$ as $n \rightarrow \infty$, where C is a positive finite constant. Typically, $C \in (0, 1)$ so it is expected that there are no observations in a data sample that are greater than the expectile at the extreme level τ'_n .
- When type="ExpectRisks", the null hypothesis of the hypothesis testing concerns the equality among the expectiles of the marginal distributions. See Section 3.3 of Padoan and Stupfler (2020) for details. When type="QuantRisks", the null hypothesis of the hypothesis testing concerns the equality among the quantiles of the marginal distributions. See Section 5 of Padoan and Stupfler (2020) for details. Note that in this case the test is based on the asymptotic distribution of normalized quantile estimator in the logarithmic scale. When type="tails", the null hypothesis of the hypothesis testing concerns the equality among the tail indices of the marginal distributions. See Sections 3.2 and 3.3 of Padoan and Stupfler (2020) for details.

- When `type="ExpectRisks"`, the null hypothesis concerns the equality among the expectiles of the marginal distributions at the intermediate level and this is achieved through `level=="inter"`. In this case the test is obtained exploiting the asymptotic distribution of relative expectile appropriately normalised. See Section 2.1, 3.1 and 3.3 of Padoan and Stupfler (2020) for details. Instead, if `level=="extreme"` the null hypothesis concerns the equality among the expectiles of the marginal distributions at the extreme level.
- When `method='LAWS'`, then the τ'_n -th d-dimensional expectile is estimated using the LAWS based estimator. When `method='QB'`, the expectile is instead estimated using the QB estimator. The definition of both estimators depend on the estimation of the d-dimensional tail index γ . The d-dimensional tail index γ is estimated using the d-dimensional Hill estimator (`tailest='Hill'`), see [MultiHTailIndex](#)). See Section 2.2 in Padoan and Stupfler (2020) for details.
- If `bias=TRUE` then d-dimensional γ is estimated using formula (4.2) of Haan et al. (2016). This is used by the LAWS and QB estimators. Furthermore, the τ'_n -th quantile is estimated using the formula in page 330 of de Haan et al. (2016). This provides a bias corrected version of the Weissman estimator. This is used by the QB estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. Instead, for simplicity the asymptotic variance-covariance matrix is estimated by the formulas Section 3.2 of Padoan and Stupfler (2020).
- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, for each marginal distribution when `tau=NULL` and `method='LAWS'` or `method='QB'`, then $\tau_n = 1 - k_n/n$ is the intermediate level of the expectile to be estimated. When `tailest='Hill'`, for each marginal distributions, then k_n specifies the number of $k+1$ larger order statistics used in the definition of the Hill estimator.
- A small value $\alpha \in (0, 1)$ specifies the significance level of Wald-type hypothesis testing.

Value

A list with elements:

- `logLikR`: the observed value of log-likelihood ratio statistic test;
- `critVal`: the quantile (critical level) of a chi-square distribution with `d` degrees of freedom and confidence level α .

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References

Padoan A.S. and Stupfler, G. (2020). Joint inference on extreme expectiles for multivariate heavy-tailed distributions. *arXiv e-prints* arXiv:2007.08944, <https://arxiv.org/abs/2007.08944>

See Also

[MultiHTailIndex](#), [predMultiExpectiles](#), [extMultiQuantile](#)

Examples

```
# Hypothesis testing on the equality extreme expectiles based on a sample of
# d-dimensional observations simulated from a joint distribution with
# a Gumbel copula and equal Frechet marginal distributions.
library(plot3D)
library(copula)
library(evd)

# distributional setting
copula <- "Gumbel"
dist <- "Frechet"

# parameter setting
dep <- 3
dim <- 3
scale <- rep(1, dim)
shape <- rep(3, dim)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.95
# Extreme level (or tail probability 1-tau1 of unobserved expectile)
tau1 <- 0.9995

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
# marginals distributions and a Gumbel copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# Performs Wald-type hypothesis testing
HypoTesting(data, tau, tau1)

# Hypothesis testing on the equality extreme expectiles based on a sample of
# d-dimensional observations simulated from a joint distribution with
# a Clayton copula and different Frechet marginal distributions.

# distributional setting
copula <- "Clayton"
dist <- "Frechet"

# parameter setting
dim <- 3
dep <- 2
scale <- rep(1, dim)
shape <- c(2.1, 3, 4.5)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.95
```

```

# Extreme level (or tail probability 1-tau1 of unobserved expectile)
tau1 <- 0.9995

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
# marginals distributions and a Gumbel copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# Performs Wald-type hypothesis testing
HypoTesting(data, tau, tau1)

```

MLTailIndex

Maximum Likelihood Tail Index Estimation

Description

Computes a point and interval estimate of the tail index based on the Maximum Likelihood (ML) estimator.

Usage

```
MLTailIndex(data, k, var=FALSE, varType="asym-Dep", bigBlock=NULL,
            smallBlock=NULL, alpha=0.05)
```

Arguments

data	A vector of $(1 \times n)$ observations.
k	An integer specifying the value of the intermediate sequence k_n . See Details .
var	If var=TRUE then an estimate of the asymptotic variance of the tail index estimator is computed.
varType	A string specifying the asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for serial dependent observations. See Details .
bigBlock	An interger specifying the size of the big-block used to estimaste the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimaste the asymptotic variance. See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the tail index.

Details

For a dataset `data` of sample size n , the tail index γ of its (marginal) distribution is computed by applying the ML estimator. The observations can be either independent or temporal dependent.

- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate γ .
- If `var=TRUE` then the asymptotic variance of the Hill estimator is computed. With independent observations the asymptotic variance is estimated by the formula in Theorem 3.4.2 of de Haan and Ferreira (2006). This is achieved through `varType="asym-Ind"`. With serial dependent observations the asymptotic variance is estimated by the formula in 1288 in Drees (2000). This is achieved through `varType="asym-Dep"`. In this latter case the serial dependence is estimated by exploiting the "big blocks separated by small blocks" technique which is a standard tool in time series, see Leadbetter et al. (1986). See also formula (11) in Drees (2003). The size of the big and small blocks are specified by the parameters `bigBlock` and `smallBlock`, respectively.
- Given a small value $\alpha \in (0, 1)$ then an asymptotic confidence interval for the tail index, with approximate nominal confidence level $(1 - \alpha)100\%$ is computed.

Value

A list with elements:

- `gammaHat`: an estimate of tail index γ ;
- `VarGamHat`: an estimate of the variance of the ML estimator;
- `CIgamHat`: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for γ .

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- Drees, H. (2000). Weighted approximations of tail processes for β -mixing random variables. *Annals of Applied Probability*, **10**, 1274-1301.
- de Haan, L. and Ferreira, A. (2006). Extreme Value Theory: An Introduction. *Springer-Verlag*, New York.
- Leadbetter, M.R., Lindgren, G. and Rootzen, H. (1989). Extremes and related properties of random sequences and processes. *Springer*.

See Also

[HTailIndex](#), [MomTailIndex](#), [EBTailIndex](#)

Examples

```

# Tail index estimation based on the Maximum Likelihood estimator obtained with
# 1-dimensional data simulated from an AR(1) with univariate Student-t
# distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# Number of larger order statistics
k <- 150

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# tail index estimation
gammaHat <- MLTailIndex(data, k, TRUE, bigBlock=bigBlock, smallBlock=smallBlock)
gammaHat$gammaHat
gammaHat$CIgamHat

```

MomTailIndex

Moment based Tail Index Estimation

Description

Computes a point estimate of the tail index based on the Moment Based (MB) estimator.

Usage

```
MomTailIndex(data, k)
```

Arguments

data A vector of $(1 \times n)$ observations.

k An integer specifying the value of the intermediate sequence k_n . See **Details**.

Details

For a dataset `data` of sample size n , the tail index γ of its (marginal) distribution is computed by applying the MB estimator. The observations can be either independent or temporal dependent. For details see de Haan and Ferreira (2006).

- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate γ .

Value

An estimate of the tail index γ .

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References

de Haan, L. and Ferreira, A. (2006). Extreme Value Theory: An Introduction. *Springer-Verlag*, New York.

See Also

[HTailIndex](#), [MLTailIndex](#), [EBTailIndex](#)

Examples

```
# Tail index estimation based on the Moment estimator obtained with
# 1-dimensional data simulated from an AR(1) with univariate Student-t
# distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallblock <- 15

# Number of larger order statistics
k <- 150

# sample size
ndata <- 2500
```

```
# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# tail index estimation
gammaHat <- MomTailIndex(data, k)
gammaHat
```

MultiHTailIndex

*Multidimensional Hill Tail Index Estimation***Description**

Computes point estimates and $(1 - \alpha)100\%$ confidence regions estimate of d -dimensional tail indices based on the Hill's estimator.

Usage

```
MultiHTailIndex(data, k, var=FALSE, varType="asym-Dep", bias=FALSE,
  alpha=0.05, plot=FALSE)
```

Arguments

data	A matrix of $(n \times d)$ observations.
k	An integer specifying the value of the intermediate sequence k_n . See Details .
var	If var=TRUE then an estimate of the variance-covariance matrix of the tail indices estimators is computed.
varType	A string specifying the asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for d dependent marginal variables. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the tail index.
plot	A logical value. By default plot=FALSE specifies that no graphical representation of the estimates is provided. See Details .

Details

For a dataset data of $(n \times d)$ observations, where d is the number of variables and n is the sample size, the tail index γ of the d marginal distributions is estimated by applying the Hill estimator. Together with a point estimate a $(1 - \alpha)100\%$ confidence region is computed. The data are regarded as d -dimensional temporal independent observations coming from dependent variables.

- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, the value k_n specifies the number of $k+1$ larger order statistics to be used to estimate each marginal tail index γ_j for $j = 1, \dots, d$.

- If `var=TRUE` then an estimate of the asymptotic variance-covariance matrix of the multivariate Hill estimator is computed. With independent observations the asymptotic variance-covariance matrix is estimated by the matrix $\hat{\Sigma}_{j,\ell}^{LAW S}(\gamma, R)(1, 1)$, see bottom formula in page 14 of Padoan and Stupfler (2020). This is achieved through `varType="asym-Dep"` which means d dependent marginal variables. When `varType="asym-Ind"` d marginal variables are regarded as independent and the returned variance-covariance matrix $\hat{\Sigma}_{j,\ell}^{LAW S}(\gamma, R)(1, 1)$ is a diagonal matrix with only variance terms.
- If `bias=TRUE` then an estimate of the bias term of the Hill estimator is computed implementing using formula (4.2) in de Haan et al. (2016). In this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.1 but instead for simplicity the formula at the bottom of page 14 in Padoan and Stupfler (2020) is still used.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence region for γ_j , for $j = 1, \dots, d$, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence intervals are computed exploiting the asymptotic normality of multivariate Hill estimator appropriately normalized (the logarithmic scale is not used), see Padoan and Stupfler (2020) for details.
- If `plot=TRUE` then a graphical representation of the estimates is not provided.

Value

A list with elements:

- `gammaHat`: an estimate of the d tail indices γ_j , for $j = 1, \dots, d$;
- `VarCovGHat`: an estimate of the asymptotic variance-covariance matrix of the multivariate Hill estimator;
- `biasTerm`: an estimate of bias term of the multivariate Hill estimator;
- `EstConReg`: an estimate of the $(1 - \alpha)100\%$ confidence region.

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- de Haan, L., Mercadier, C. and Zhou, C. (2016). Adapting extreme value statistics to financial time series: dealing with bias and serial dependence. *Finance and Stochastics*, **20**, 321-354.
- de Haan, L. and Ferreira, A. (2006). Extreme Value Theory: An Introduction. *Springer-Verlag*, New York.

See Also

[HTailIndex](#), [rmdata](#)

Examples

```

# Tail index estimation based on the multivariate Hill estimator obtained with
# n observations simulated from a d-dimensional random vector with a multivariate
# distribution with equal Frechet margins and a Clayton copula.
library(plot3D)
library(copula)
library(efd)

# distributional setting
copula <- "Clayton"
dist <- "Frechet"

# parameter setting
dep <- 3
dim <- 3
scale <- rep(1, dim)
shape <- rep(3, dim)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# Number of larger order statistics
k <- 150

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
# marginals distributions and a Clayton copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# tail indices estimation
est <- MultiHTailIndex(data, k, TRUE)
est$gammaHat
est$VarCovGHat
# run the following command to see the graphical representation

est <- MultiHTailIndex(data, k, TRUE, plot=TRUE)

```

predExpectiles

Extreme Expectile Estimation

Description

Computes a point and interval estimate of the expectile at the extreme level (Expectile Prediction).

Usage

```

predExpectiles(data, tau, tau1, method="LAWS", tailest="Hill", var=FALSE,
               varType="asym-Dep", bias=FALSE, bigBlock=NULL, smallBlock=NULL,
               k=NULL, alpha_n=NULL, alpha=0.05)

```

Arguments

data	A vector of $(1 \times n)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
method	A string specifying the method used to estimate the expectile. By default <code>est="LAWS"</code> specifies the use of the LAWS based estimator. See Details .
tailest	A string specifying the tail index estimator. By default <code>tailest="Hill"</code> specifies the use of Hill estimator. See Details .
var	If <code>var=TRUE</code> then an estimate of the asymptotic variance of the expectile estimator is computed.
varType	A string specifying the type of asymptotic variance to compute. By default <code>varType="asym-Dep"</code> specifies the variance estimator for serial dependent observations. See Details .
bias	A logical value. By default <code>bias=FALSE</code> specifies that no bias correction is computed. See Details .
bigBlock	An interger specifying the size of the big-block used to estimate the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimate the asymptotic variance. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha_n	A real in $(0, 1)$ specifying the quantile's extreme level to be use in order to estimate the expectile's extreme level.
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the expectile at the intermedite level.

Details

For a dataset `data` of sample size n , an estimate of the τ'_n -th expectile is computed. The estimation of the expectile at the extreme level `tau1` (τ'_n) is meant to be a prediction beyond the observed sample. Two estimators are available: the so-called Least Asymmetrically Weighted Squares (LAWS) based estimator and the Quantile-Based (QB) estimator. The definition of both estimators depends on the estimation of the tail index γ . Here, γ is estimated using the Hill estimation (see [HTailIndex](#) for details) or in alternative using the the expectile based estimator (see [EBTailIndex](#)). The observations can be either independent or temporal dependent. See Section 3.2 in Padoan and Stupfler (2020) for details.

- The so-called intermediate level `tau` or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, $\tau_n \in (0, 1)$ is the ratio between N (Numerator) and D (Denominator). Where N is the empirical mean distance of the τ_n -th expectile from the observations smaller than it, and D is the empirical mean distance of τ_n -th expectile from all the observations.
- The so-called extreme level `tau1` or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. The value $(1 - \tau'_n) \in (0, 1)$ is meant to be a small tail probability such that $(1 - \tau'_n) = 1/n$ or $(1 - \tau'_n) < 1/n$. It is also assumed that $n(1 - \tau'_n) \rightarrow C$ as $n \rightarrow \infty$, where C is a positive finite constant. Typically, $C \in (0, 1)$ so it is expected that there are no observations in a data sample that are greater than the expectile at the extreme level τ'_n .

- When method='LAWS', then the τ'_n -th expectile is estimated using the LAWS based estimator. When method='QB', the expectile is instead estimated using the QB estimator. The definition of both estimators depend on the estimation of the tail index γ . When tailest='Hill' then γ is estimated using the Hill estimator (see [HTailIndex](#)). When tailest='ExpBased', then γ is estimated using the expectile based estimator (see [EBTailIndex](#)). See Section 3.2 in Padoan and Stupfler (2020) for details.
- If var=TRUE then an estimate of the asymptotic variance of the τ'_n -th expectile is computed. Notice that the estimation of the asymptotic variance **is only available** when γ is estimated using the Hill estimator (see [HTailIndex](#)). With independent observations the asymptotic variance is estimated by $\hat{\gamma}^2$, see the remark below Theorem 3.5 in Padoan and Stupfler (2020). This is achieved through varType="asym-Ind". With serial dependent observations the asymptotic variance is estimated by the formula in Theorem 3.5 of Padoan and Stupfler (2020). This is achieved through varType="asym-Dep". See Section 3.2 in Padoan and Stupfler (2020) for details. In this latter case the computation of the serial dependence is based on the "big blocks separated by small blocks" technique which is a standard tool in time series, see e.g. Leadbetter et al. (1986). The size of the big and small blocks are specified by the parameters bigBlock and smallBlock, respectively.
- If bias=TRUE then γ is estimated using formula (4.2) of Haan et al. (2016). This is used by the LAWS and QB estimators. Furthermore, the τ'_n -th quantile is estimated using the formula in page 330 of de Haan et al. (2016). This provides a bias corrected version of the Weissman estimator. This is used by the QB estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. Instead, for simplicity the asymptotic variance is estimated by the formula in Corollary 3.8, with serial dependent observations, and $\hat{\gamma}^2$ with independent observation (see e.g. de Drees 2000, for the details).
- k or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, when tau=NULL and method='LAWS', then $\tau_n = 1 - k_n/n$ is the intermediate level of the expectile to be estimated. The latter is also used to estimate the tail index when tailest='ExpBased'. Instead, if tailest='Hill', then k_n specifies the number of $k+1$ larger order statistics used in the definition of the Hill estimator. Differently, When tau=NULL and method='QB', then $\tau_n = 1 - k_n/n$ is the intermediate level of the quantile to be estimated and of the expectile to be estimated when tailest='ExpBased'. Instead, when tailest='Hill' it is the number of $k+1$ larger order statistics used in the definition of the Hill estimator.
- If quantile's extreme level is provided by alpha_n, then expectile's extreme level $\tau'_n(\alpha_n)$ is replaced by $\tau'_n(\alpha_n)$ which is estimated using the method described in Section 6 of Padoan and Stupfler (2020). See [estExtLevel](#) for details.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence interval for τ'_n -th expectile, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence intervals are computed exploiting formula (10) and (11) in Padoan and Stupfler (2020) and (46) in Drees (2003). See Section 5 in Padoan and Stupfler (2020) for details. When bias=TRUE confidence intervals are computed in the same way but after correcting the tail index estimate by an estimate of the bias term, see formula (4.2) in de Haan et al. (2016) for details.

Value

A list with elements:

- EExpHat: an estimate of the τ'_n -th expectile;
- VarExtHat: an estimate of the asymptotic variance of the expectile estimator;
- CIExpct: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for τ'_n -th expectile.

Author(s)

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References

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- Drees, H. (2003). Extreme quantile estimation for dependent data, with applications to finance. *Bernoulli*, **9**, 617-657.
- Drees, H. (2000). Weighted approximations of tail processes for β -mixing random variables. *Annals of Applied Probability*, **10**, 1274-1301.
- Leadbetter, M.R., Lindgren, G. and Rootzen, H. (1989). Extremes and related properties of random sequences and processes. *Springer*.

See Also

[HTailIndex](#), [EBTailIndex](#), [estExpectiles](#), [extQuantile](#)

Examples

```
# Extreme expectile estimation at the extreme level tau1 obtained with
# 1-dimensional data simulated from an AR(1) with univariate
# Student-t distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.95
```

```

# Extreme level (or tail probability 1-tau1 of unobserved expectile)
tau1 <- 0.9995

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# Extreme expectile estimation
expectHat1 <- predExpectiles(data, tau, tau1, var=TRUE, bigBlock=bigBlock,
                             smallBlock=smallBlock)
expectHat1$EExpHat
expectHat1$CIEpct
# Extreme expectile estimation with bias correction
tau <- 0.80
expectHat2 <- predExpectiles(data, tau, tau1, "QB", var=TRUE, bias=TRUE, bigBlock=bigBlock,
                             smallBlock=smallBlock)
expectHat2$EExpHat
expectHat2$CIEpct

```

predMultiExpectiles *Multidimensional Extreme Expectile Estimation*

Description

Computes point estimates and $(1 - \alpha)100\%$ confidence regions for d -dimensional expectile at the extreme level (Expectile Prediction).

Usage

```

predMultiExpectiles(data, tau, tau1, method="LAWS", tailest="Hill", var=FALSE,
                    varType="asym-Ind-Adj-Log", bias=FALSE, k=NULL, alpha=0.05,
                    plot=FALSE)

```

Arguments

data	A matrix of $(n \times d)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
method	A string specifying the method used to estimate the expectile. By default <code>est="LAWS"</code> specifies the use of the LAWS based estimator. See Details .
tailest	A string specifying the tail index estimator. By default <code>tailest="Hill"</code> specifies the use of Hill estimator. See Details .
var	If <code>var=TRUE</code> then an estimate of the asymptotic variance of the expectile estimator is computed.

varType	A string specifying the type of asymptotic variance-covariance matrix to compute. By default varType="asym-Ind-Adj-Log" specifies that the variance-covariance matrix is computed assuming dependent variables and exploiting the log scale and a suitable adjustment. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence region for the d-dimensional expectile at the extreme level.
plot	A logical value. By default plot=FALSE specifies that no graphical representation of the estimates is provided. See Details .

Details

For a dataset data of d-dimensional observations and sample size n , an estimate of the τ'_n -th d-dimensional expectile is computed. The estimation of the d-dimensional expectile at the extreme level tau1 (τ'_n) is meant to be a prediction beyond the observed sample. Two estimators are available: the so-called Least Asymmetrically Weighted Squares (LAWS) based estimator and the Quantile-Based (QB) estimator. The definition of both estimators depends on the estimation of the d-dimensional tail index γ . Here, γ is estimated using the Hill estimation (see [MultiHTailIndex](#) for details). The data are regarded as d-dimensional temporal independent observations coming from dependent variables. See Padoan and Stupfler (2020) for details.

- The so-called intermediate level tau or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. Practically, for each marginal distribution, $\tau_n \in (0, 1)$ is the ratio between N (Numerator) and D (Denominator). Where N is the empirical mean distance of the τ_n -th expectile from the observations smaller than it, and D is the empirical mean distance of τ_n -th expectile from all the observations.
- The so-called extreme level tau1 or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. For each marginal distribution, the value $(1 - \tau'_n) \in (0, 1)$ is meant to be a small tail probability such that $(1 - \tau'_n) = 1/n$ or $(1 - \tau'_n) < 1/n$. It is also assumed that $n(1 - \tau'_n) \rightarrow C$ as $n \rightarrow \infty$, where C is a positive finite constant. Typically, $C \in (0, 1)$ so it is expected that there are no observations in a data sample that are greater than the expectile at the extreme level τ'_n .
- When method='LAWS', then the τ'_n -th d-dimensional expectile is estimated using the LAWS based estimator. When method='QB', the expectile is instead estimated using the QB estimator. The definition of both estimators depend on the estimation of the d-dimensional tail index γ . The d-dimensional tail index γ is estimated using the d-dimensional Hill estimator (tailest='Hill'), see [MultiHTailIndex](#). This is the only available option so far (soon more results will be available). See Section 2.2 in Padoan and Stupfler (2020) for details.
- If var=TRUE then an estimate of the asymptotic variance-covariance matrix of the τ'_n -th d-dimensional expectile is computed. Notice that the estimation of the asymptotic variance-covariance matrix **is only available** when γ is estimated using the Hill estimator (see [MultiHTailIndex](#)). The data are regarded as temporal independent observations coming from dependent variables. The asymptotic variance-covariance matrix is estimated exploiting the formulas in Section 3.2 of Padoan and Stupfler (2020). The variance-covariance matrix is computed exploiting the asymptotic behaviour of the normalized expectile estimator which is expressed

in logarithmic scale. In addition, a suitable adjustment is considered. This is achieved through `varType="asym-Ind-Adj-Log"`. The data can also be regarded as coded-dimensional temporal independent observations coming from independent variables. In this case the asymptotic variance-covariance matrix is diagonal and is also computed exploiting the formulas in Section 3.2 of Padoan and Stupfler (2020). This is achieved through `varType="asym-Ind-Log"`. If `varType="asym-Ind-Adj"`, then the variance-covariance matrix is computed exploiting the asymptotic behaviour of the relative expectile estimator appropriately normalized and exploiting a suitable adjustment. This concerns the case of dependent variables. The case of independent variables is achieved through `varType="asym-Ind"`.

- If `bias=TRUE` then d -dimensional γ is estimated using formula (4.2) of Haan et al. (2016). This is used by the LAWS and QB estimators. Furthermore, the τ'_n -th quantile is estimated using the formula in page 330 of de Haan et al. (2016). This provides a bias corrected version of the Weissman estimator. This is used by the QB estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. Instead, for simplicity the asymptotic variance-covariance matrix is estimated by the formulas Section 3.2 of Padoan and Stupfler (2020).
- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Practically, for each marginal distribution when `tau=NULL` and `method='LAWS'` or `method='QB'`, then $\tau_n = 1 - k_n/n$ is the intermediate level of the expectile to be estimated. When `tailest='Hill'`, for each marginal distributions, then k_n specifies the number of $k+1$ larger order statistics used in the definition of the Hill estimator.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence region for τ'_n -th d -dimensional expectile, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence regions are computed exploiting the formulas in Section 3.2 of Padoan and Stupfler (2020). If `varType="asym-Ind-Adj-Log"`, then an "asymmetric" confidence regions is computed exploiting the asymptotic behaviour of the normalized expectile estimator in logarithmic scale and using a suitable adjustment. This choice is recommended. If `varType="asym-Ind-Adj"`, then the a "symmetric" confidence regions is computed exploiting the asymptotic behaviour of the relative expectile estimator appropriately normalized.
- If `plot=TRUE` then a graphical representation of the estimates is not provided.

Value

A list with elements:

- `ExpctHat`: an estimate of the τ'_n -th d -dimensional expectile;
- `biasTerm`: an estimate of the bias term of $y_{j\tau'_n}$ -th d -dimensional expectile;
- `VarCovEHat`: an estimate of the asymptotic variance-covariance of the d -dimensional expectile estimator;
- `EstConReg`: an estimate of the approximate $(1 - \alpha)100\%$ confidence regions for τ'_n -th d -dimensional expectile.

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References

Padoan A.S. and Stupfler, G. (2020). Joint inference on extreme expectiles for multivariate heavy-tailed distributions. *arXiv e-prints* arXiv:2007.08944, <https://arxiv.org/abs/2007.08944>

See Also

[MultiHTailIndex](#), [estMultiExpectiles](#), [extMultiQuantile](#)

Examples

```
# Extreme expectile estimation at the extreme level tau1 obtained with
# d-dimensional observations simulated from a joint distribution with
# a Gumbel copula and equal Frechet marginal distributions.
library(plot3D)
library(copula)
library(evd)

# distributional setting
copula <- "Gumbel"
dist <- "Frechet"

# parameter setting
dep <- 3
dim <- 3
scale <- rep(1, dim)
shape <- rep(3, dim)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# Intermediate level (or sample tail probability 1-tau)
tau <- 0.95
# Extreme level (or tail probability 1-tau1 of unobserved expectile)
tau1 <- 0.9995

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
# marginals distributions and a Gumbel copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# High d-dimensional expectile (intermediate level) estimation
expectHat <- predMultiExpectiles(data, tau, tau1, var=TRUE)

expectHat$ExpctHat
expectHat$VarCovEHat
# run the following command to see the graphical representation

expectHat <- predMultiExpectiles(data, tau, tau1, var=TRUE, plot=TRUE)
```

QuantMES

*Marginal Expected Shortfall Quantile Based Estimation***Description**

Computes a point and interval estimate of the Marginal Expected Shortfall (MES) using a quantile based approach.

Usage

```
QuantMES(data, tau, tau1, var=FALSE, varType="asym-Dep", bias=FALSE, bigBlock=NULL,
          smallBlock=NULL, k=NULL, alpha=0.05)
```

Arguments

data	A vector of $(1 \times n)$ observations.
tau	A real in $(0, 1)$ specifying the intermediate level τ_n . See Details .
tau1	A real in $(0, 1)$ specifying the extreme level τ'_n . See Details .
var	If var=TRUE then an estimate of the asymptotic variance of the MES estimator is computed.
varType	A string specifying the type of asymptotic variance to compute. By default varType="asym-Dep" specifies the variance estimator for serial dependent observations. See Details .
bias	A logical value. By default bias=FALSE specifies that no bias correction is computed. See Details .
bigBlock	An interger specifying the size of the big-block used to estimate the asymptotic variance. See Details .
smallBlock	An interger specifying the size of the small-block used to estimate the asymptotic variance. See Details .
k	An integer specifying the value of the intermediate sequence k_n . See Details .
alpha	A real in $(0, 1)$ specifying the confidence level $(1 - \alpha)100\%$ of the approximate confidence interval for the expectile at the intermediate level.

Details

For a dataset data of sample size n , an estimate of the τ'_n -th MES is computed. The estimation of the MES at the extreme level tau1 (τ'_n) is indeed meant to be a prediction. Estimates are obtained through the quantile based estimator defined in page 12 of Padoan and Stupfler (2020). Such an estimator depends on the estimation of the tail index γ . Here, γ is estimated using the Hill estimation (see [HTailIndex](#) for details). The observations can be either independent or temporal dependent. See Section 4 in Padoan and Stupfler (2020) for details.

- The so-called intermediate level tau or τ_n is a sequence of positive reals such that $\tau_n \rightarrow 1$ as $n \rightarrow \infty$. See [predExpectiles](#) for details.

- The so-called extreme level τ_1 or τ'_n is a sequence of positive reals such that $\tau'_n \rightarrow 1$ as $n \rightarrow \infty$. See [predExpectiles](#) for details.
- If `var=TRUE` then an estimate of the asymptotic variance of the τ'_n -th MES is computed. Notice that the estimation of the asymptotic variance is **only available** when γ is estimated using the Hill estimator (see [HTailIndex](#)). With independent observations the asymptotic variance is estimated by $\hat{\gamma}^2$, see Corollary 4.3 in Padoan and Stupfler (2020). This is achieved through `varType="asym-Ind"`. With serial dependent observations the asymptotic variance is estimated by the formula in Corollary 4.2 of Padoan and Stupfler (2020). This is achieved through `varType="asym-Dep"`. See Section 4 and 5 in Padoan and Stupfler (2020) for details. In this latter case the computation of the serial dependence is based on the "big blocks separated by small blocks" technique which is a standard tool in time series, see e.g. Leadbetter et al. (1986). The size of the big and small blocks are specified by the parameters `bigBlock` and `smallBlock`, respectively.
- If `bias=TRUE` then γ is estimated using formula (4.2) of Haan et al. (2016). This is used by the LAWS and QB estimators. Furthermore, the τ'_n -th quantile is estimated using the formula in page 330 of de Haan et al. (2016). This provides a bias corrected version of the Weissman estimator. This is used by the QB estimator. However, in this case the asymptotic variance is not estimated using the formula in Haan et al. (2016) Theorem 4.2. Instead, for simplicity the asymptotic variance is estimated by the formula in Corollary 3.8, with serial dependent observations, and $\hat{\gamma}^2$ with independent observation (see e.g. de Drees 2000, for the details).
- `k` or k_n is the value of the so-called intermediate sequence k_n , $n = 1, 2, \dots$. It represents a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. k_n specifies the number of $k+1$ larger order statistics used in the definition of the Hill estimator (see [HTailIndex](#) for details).
- If the quantile's extreme level is provided by `alpha_n`, then expectile's extreme level τ'_n is replaced by $\tau'_n(\alpha_n)$ which is estimated by the method described in Section 6 of Padoan and Stupfler (2020). See [estExtLevel](#) for details.
- Given a small value $\alpha \in (0, 1)$ then an estimate of an asymptotic confidence interval for τ'_n -th expectile, with approximate nominal confidence level $(1 - \alpha)100\%$, is computed. The confidence intervals are computed exploiting formula in Corollary 4.2, Theorem 6.2 of Padoan and Stupfler (2020) and (46) in Drees (2003). See Sections 4-6 in Padoan and Stupfler (2020) for details. When `biast=TRUE` confidence intervals are computed in the same way but after correcting the tail index estimate by an estimate of the bias term, see formula (4.2) in de Haan et al. (2016) for details.

Value

A list with elements:

- `HatQMES`: an estimate of the τ'_n -th quantile based MES;
- `VarHatQMES`: an estimate of the asymptotic variance of the quantile based MES estimator;
- `CIHatQMES`: an estimate of the approximate $(1 - \alpha)100\%$ confidence interval for τ'_n -th MES.

Author(s)

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- Drees, H. (2003). Extreme quantile estimation for dependent data, with applications to finance. *Bernoulli*, **9**, 617-657.
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- Leadbetter, M.R., Lindgren, G. and Rootzen, H. (1989). Extremes and related properties of random sequences and processes. *Springer*.

See Also

[ExpectMES](#), [HTailIndex](#), [predExpectiles](#), [extQuantile](#)

Examples

```
# Marginl Expected Shortfall quantile based estimation at the extreme level
# obtained with 2-dimensional data simulated from an AR(1) with bivariate
# Student-t distributed innovations

tsDist <- "AStudentT"
tsType <- "AR"
tsCopula <- "studentT"

# parameter setting
corr <- 0.8
dep <- 0.8
df <- 3
par <- list(corr=corr, dep=dep, df=df)

# Big- small-blocks setting
bigBlock <- 65
smallBlock <- 15

# quantile's extreme level
tau1 <- 0.9995

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rbtimeseries(ndata, tsDist, tsType, tsCopula, par)
```

```
# Extreme MES expectile based estimation
MESHat <- QuantMES(data, NULL, tau1, var=TRUE, k=150, bigBlock=bigBlock,
                  smallBlock=smallBlock)
MESHat
```

rbtimeseries

Simulation of Two-Dimensional Temporally Dependent Observations

Description

Simulates samples from parametric families of bivariate time series models.

Usage

```
rbtimeseries(ndata, dist="studentT", type="AR", copula="Gumbel", par, burnin=1e+03)
```

Arguments

ndata	A positive interger specifying the number of observations to simulate.
dist	A string specifying the parametric family of the innovations distribution. By default <code>dist="studentT"</code> specifies a Student- <i>t</i> family of distributions. See Details .
type	A string specifying the type of time series. By default <code>type="AR"</code> specifies a linear Auto-Regressive time series. See Details .
copula	A string specifying the type copula to be used. By default <code>copula="Gumbel"</code> specifies the Gumbel copula. See Details .
par	A list of p parameters to be specified for the bivariate time series parametric family. See Details .
burnin	A positive interger specifying the number of initial observations to discard from the simulated sample.

Details

For a time series class (`type`), with a parametric family (`dist`) for the innovations, a sample of size `ndata` is simulated. See for example Brockwell and Davis (2016).

- The available categories of bivariate time series models are: Auto-Regressive (`type="AR"`), Auto-Regressive and Moving-Average (`type="ARMA"`), Generalized-Autoregressive-Conditional-Heteroskedasticity (`type="GARCH"`) and Auto-Regressive.
- With AR(1) times series the available families of distributions for the innovations and the dependence structure (`copula`) are:
 - Student-*t* (`dist="studentT"` and `copula="studentT"`) with marginal parameters (equal for both distributions): $\phi \in (-1, 1)$ (autoregressive coefficient), $\nu > 0$ (degrees of freedom) and dependence parameter $dep \in (-1, 1)$. The parameters are specified as `par <-list(corr,df,dep)`;

- Asymmetric Student- t (`dist="AStudentT"` and `copula="studentT"`) with marginal parameters (equal for both distributions): $\phi \in (-1, 1)$ (autoregressive coefficient), $\nu > 0$ (degrees of freedom) and dependence parameter $dep \in (-1, 1)$. The parameters are specified as `par <-list(corr, df, dep)`. Note that in this case the tail index of the lower and upper tail of the first marginal are different, see Padoan and Stupfler (2020) for details;
- With ARMA(1,1) times series the available families of distributions for the innovations and the dependence structure (copula) are:
 - symmetric Pareto (`dist="double-Pareto"` and `copula="Gumbel"` or `copula="Gaussian"`) with marginal parameters (equal for both distributions): $\phi \in (-1, 1)$ (autoregressive coefficient), $\sigma > 0$ (scale), $\alpha > 0$ (shape), θ (movingaverage coefficient), and dependence parameter dep ($dep > 0$ if `copula="Gumbel"` or $dep \in (-1, 1)$ if `copula="Gaussian"`). The parameters are specified as `par <-list(corr, scale, shape, smooth, dep)`.
 - symmetric Pareto (`dist="double-Pareto"` and `copula="Gumbel"` or `copula="Gaussian"`) with marginal parameters (equal for both distributions): $\phi \in (-1, 1)$ (autoregressive coefficient), $\sigma > 0$ (scale), $\alpha > 0$ (shape), θ (movingaverage coefficient), and dependence parameter dep ($dep > 0$ if `copula="Gumbel"` or $dep \in (-1, 1)$ if `copula="Gaussian"`). The parameters are specified as `par <-list(corr, scale, shape, smooth, dep)`. Note that in this case the tail index of the lower and upper tail of the first marginal are different, see Padoan and Stupfler (2020) for details;
- With ARCH(1)/GARCH(1,1) time series the distribution of the innovations are symmetric Gaussian (`dist="Gaussian"`) or asymmetric Gaussian `dist="AGaussian"`. In both cases the marginal parameters (equal for both distributions) are: $\alpha_0, \alpha_1, \beta$. In the asymmetric Gaussian case the tail index of the lower and upper tail of the first marginal are different, see Padoan and Stupfler (2020) for details. The available copulas are: Gaussian (`copula="Gaussian"`) with dependence parameter $dep \in (-1, 1)$, Student- t (`copula="studentT"`) with dependence parameters $dep \in (-1, 1)$ and $\nu > 0$ (degrees of freedom), Gumbel (`copula="Gumbel"`) with dependence parameter $dep > 0$. The parameters are specified as `par <-list(alpha0, alpha1, beta, dep)` or `par <-list(alpha0, alpha1, beta, dep, df)`.

Value

A vector of $(2 \times n)$ observations simulated from a specified bivariate time series model.

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References

- Brockwell, Peter J., and Richard A. Davis. (2016). Introduction to time series and forecasting. *Springer*.
- Padoan A.S. and Stupfler, G. (2020). Extreme expectile estimation for heavy-tailed time series. *arXiv e-prints* arXiv:2004.04078, <https://arxiv.org/abs/2004.04078>.

See Also

[rtimeseries](#), [expectiles](#)

Examples

```
# Data simulation from a 2-dimensional AR(1) with bivariate Student-t distributed
# innovations, with one marginal distribution whose lower and upper tail indices
# that are different

tsDist <- "AStudentT"
tsType <- "AR"
tsCopula <- "studentT"

# parameter setting
corr <- 0.8
dep <- 0.8
df <- 3
par <- list(corr=corr, dep=dep, df=df)

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rbtimeseries(ndata, tsDist, tsType, tsCopula, par)

# Extreme expectile estimation
plot(data, pch=21)
plot(data[,1], type="l")
plot(data[,2], type="l")
```

rmdata

Simulation of d -Dimensional Temporally Independent Observations

Description

Simulates samples of independent d -dimensional observations from parametric families of joint distributions with a given copula and equal marginal distributions.

Usage

```
rmdata (ndata, dist="studentT", copula="studentT", par)
```

Arguments

ndata	A positive interger specifying the number of observations to simulate.
dist	A string specifying the parametric family of equal marginal distributions. By default <code>dist="studentT"</code> specifies a Student- t family of distributions. See Details .
copula	A string specifying the type copula to be used. By default <code>copula="studentT"</code> specifies the Student- t copula. See Details .
par	A list of p parameters to be specified for the multivariate parametric family of distributions. See Details .

Details

For a joint multivariate distribution with a given parametric copula class (copula) and a given parametric family of equal marginal distributions (dist), a sample of size ndata is simulated.

- The available copula classes are: Student-*t* (copula="studentT") with $\nu > 0$ degrees of freedom (df) and scale parameters $\rho_{i,j} \in (-1, 1)$ for $i \neq j = 1, \dots, d$ (sigma), Gaussian (copula="Gaussian") with correlation parameters $\rho_{i,j} \in (-1, 1)$ for $i \neq j = 1, \dots, d$ (sigma), Clayton (copula="Clayton") with dependence parameter $\theta > 0$ (dep), Gumbel (copula="Gumbel") with dependence parameter $\theta \geq 1$ (dep) and Frank (copula="Frank") with dependence parameter $\theta > 0$ (dep).
- The available families of marginal distributions are:
 - Student-*t* (dist="studentT") with $\nu > 0$ degrees of freedom (df);
 - Asymmetric Student-*t* (dist="AstudentT") with $\nu > 0$ degrees of freedom (df). In this case all the observations are only positive;
 - Frechet (dist="Frechet") with scale $\sigma > 0$ (scale) and shape $\alpha > 0$ (shape) parameters.
 - Frechet (dist="double-Frechet") with scale $\sigma > 0$ (scale) and shape $\alpha > 0$ (shape) parameters. In this case positive and negative observations are allowed;
 - symmetric Pareto (dist="double-Pareto") with scale $\sigma > 0$ (scale) and shape $\alpha > 0$ (shape) parameters. In this case positive and negative observations are allowed.
- The available classes of multivariate joint distributions are:
 - studentT-studentT (dist="studentT" and copula="studentT") with parameters par <-list(df, sigma);
 - studentT (dist="studentT" and copula="None" with parameters par <-list(df, dim). In this case the d variables are regarded as independent;
 - studentT-AstudentT (dist="AstudentT" and copula="studentT") with parameters par <-list(df, sigma, shape);
 - Gaussian-studentT (dist="studentT" and copula="Gaussian") with parameters par <-list(df, sigma);
 - Gaussian-AstudentT (dist="AstudentT" and copula="Gaussian") with parameters par <-list(df, sigma, shape);
 - Frechet (dist="Frechet" and copula="None") with parameters par <-list(shape, dim). In this case the d variables are regarded as independent;
 - Clayton-Frechet (dist="Frechet" and copula="Clayton") with parameters par <-list(dep, dim, scale, shape);
 - Gumbel-Frechet (dist="Frechet" and copula="Gumbel") with parameters par <-list(dep, dim, scale, shape);
 - Frank-Frechet (dist="Frechet" and copula="Frank") with parameters par <-list(dep, dim, scale, shape);
 - Clayton-double-Frechet (dist="double-Frechet" and copula="Clayton") with parameters par <-list(dep, dim, scale, shape);
 - Gumbel-double-Frechet (dist="double-Frechet" and copula="Gumbel") with parameters par <-list(dep, dim, scale, shape);
 - Frank-double-Frechet (dist="double-Frechet" and copula="Frank") with parameters par <-list(dep, dim, scale, shape);
 - Clayton-double-Pareto (dist="double-Pareto" and copula="Clayton") with parameters par <-list(dep, dim, scale, shape);

- Gumbel-double-Pareto (`dist="double-Pareto"` and `copula="Gumbel"`) with parameters `par <-list(dep,dim,scale,shape)`;
- Frank-double-Pareto (`dist="double-Pareto"` and `copula="Frank"`) with parameters `par <-list(dep,dim,scale,shape)`.

Note that above `dim` indicates the number of d marginal variables.

Value

A matrix of $(n \times d)$ observations simulated from a specified multivariate parametric joint distribution.

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References

- Joe, H. (2014). Dependence Modeling with Copulas. Chapman & Hall/CRC Press, Boca Raton, USA.
- Padoan A.S. and Stupfler, G. (2020). Joint inference on extreme expectiles for multivariate heavy-tailed distributions. *arXiv e-prints* arXiv:2007.08944, <https://arxiv.org/abs/2007.08944>.

See Also

[rtimeseries](#), [rbtimeseries](#)

Examples

```
library(plot3D)
library(copula)
library(evd)

# Data simulation from a 3-dimensional random vector a with multivariate distribution
# given by a Gumbel copula and three equal Frechet marginal distributions

# distributional setting
copula <- "Gumbel"
dist <- "Frechet"

# parameter setting
dep <- 3
dim <- 3
scale <- rep(1, dim)
shape <- rep(3, dim)
par <- list(dep=dep, scale=scale, shape=shape, dim=dim)

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Frechet
```

```

# marginal distributions and a Gumbel copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

# Data simulation from a 3-dimensional random vector a with multivariate distribution
# given by a Gaussian copula and three equal Student-t marginal distributions

# distributional setting
dist <- "studentT"
copula <- "Gaussian"

# parameter setting
rho <- c(0.9, 0.8, 0.7)
sigma <- c(1, 1, 1)
Sigma <- sigma^2 * diag(dim)
Sigma[lower.tri(Sigma)] <- rho
Sigma <- t(Sigma)
Sigma[lower.tri(Sigma)] <- rho
df <- 3
par <- list(sigma=Sigma, df=df)

# sample size
ndata <- 1000

# Simulates a sample from a multivariate distribution with equal Student-t
# marginal distributions and a Gaussian copula
data <- rmdata(ndata, dist, copula, par)
scatter3D(data[,1], data[,2], data[,3])

```

rtimeseries

Simulation of One-Dimensional Temporally Dependent Observations

Description

Simulates samples from parametric families of time series models.

Usage

```
rtimeseries(ndata, dist="studentT", type="AR", par, burnin=1e+03)
```

Arguments

ndata	A positive interger specifying the number of observations to simulate.
dist	A string specifying the parametric family of the innovations distribution. By default <code>dist="studentT"</code> specifies a Student- <i>t</i> family of distributions. See Details .

type	A string specifying the type of time series. By default type="AR" specifies a linear Auto-Regressive time series. See Details .
par	A vector of $(1 \times p)$ parameters to be specified for the univariate time series parametric family. See Details .
burnin	A positive interger specifying the number of initial observations to discard from the simulated sample.

Details

For a time series class (type) with a parametric family (dist) for the innovations, a sample of size ndata is simulated. See for example Brockwell and Davis (2016).

- The available categories of time series models are: Auto-Regressive (type="AR"), Auto-Regressive and Moving-Average (type="ARMA"), Generalized-Autoregressive-Conditional-Heteroskedasticity (type="GARCH") and Auto-Regressive and Moving-Maxima (type="ARMAX").
- With AR(1) and ARMA(1,1) times series the available families of distributions for the innovations are:
 - Student- t (dist="studentT") with parameters: $\phi \in (-1, 1)$ (autoregressive coefficient), $\nu > 0$ (degrees of freedom) specified by par=c(corr, df);
 - symmetric Frechet (dist="double-Frechet") with parameters $\phi \in (-1, 1)$ (autoregressive coefficient), $\sigma > 0$ (scale), $\alpha > 0$ (shape), θ (movingaverage coefficient), specified by par=c(corr, scale, shape, smooth);
 - symmetric Pareto (dist="double-Pareto") with parameters $\phi \in (-1, 1)$ (autoregressive coefficient), $\sigma > 0$ (scale), $\alpha > 0$ (shape), θ (movingaverage coefficient), specified by par=c(corr, scale, shape, smooth).

With ARCH(1)/GARCH(1,1) time series the Gaussian family of distributions is available for the innovations (dist="Gaussian") with parameters, $\alpha_0, \alpha_1, \beta$ specified by par=c(alpha0, alpha1, beta). Finally, with ARMAX(1) times series the Frechet families of distributions is available for the innovations (dist="Frechet") with parameters, $\phi \in (-1, 1)$ (autoregressive coefficient), $\sigma > 0$ (scale), $\alpha > 0$ (shape) specified by par=c(corr, scale, shape).

Value

A vector of $(1 \times n)$ observations simulated from a specified time series model.

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References

- Brockwell, Peter J., and Richard A. Davis. (2016). Introduction to time series and forecasting. *Springer*.
- Padoan A.S. and Stupfler, G. (2020). Extreme expectile estimation for heavy-tailed time series. *arXiv e-prints* arXiv:2004.04078, <https://arxiv.org/abs/2004.04078>.

See Also

[expectiles](#)

Examples

```
# Data simulation from a 1-dimensional AR(1) with univariate Student-t
# distributed innovations

tsDist <- "studentT"
tsType <- "AR"

# parameter setting
corr <- 0.8
df <- 3
par <- c(corr, df)

# sample size
ndata <- 2500

# Simulates a sample from an AR(1) model with Student-t innovations
data <- rtimeseries(ndata, tsDist, tsType, par)

# Graphic representation
plot(data, type="l")
acf(data)
```

 sp500

Negative log-returns of S&P 500.

Description

Series of negative log-returns of the U.S. stock market index Standard and Poor 500.

Format

A 8784 * 2 data frame.

Details

From the series of $n = 8785$ closing prices S_t , $t = 1, 2, \dots$, for the Standard and Poor 500 stock market index, recorded from January 29, 1985 to December 12, 2019, the series of negative log-returns.

$$X_{t+1} = -\log(S_{t+1}/S_t), \quad 1 \leq t \leq n - 1$$

is available. Hence the dataset (negative log-returns) contains 8784 observations.

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